UNIVERSITI SAINS MALAYSIA

Final Examination
2015/2016 Academic Session

May/June 2016

JIM 211 – Advanced Calculus
[Kalkulus Lanjutan]

Duration : 3 hours
[*Masa: 3 jam*]

Please ensure that this examination paper contains **EIGHT** printed pages before you begin the examination.

Answer **ALL** questions. You may answer either in Bahasa Malaysia or in English.

Read the instructions carefully before answering.

Each question is worth 100 marks.

In the event of any discrepancies, the English version shall be used.

*[Sila pastikan bahawa kertas peperiksaan ini mengandungi **LAPAN** muka surat yang bercetak sebelum anda memulakan peperiksaan ini.*

*Jawab **SEMUA** soalan. Anda dibenarkan menjawab sama ada dalam Bahasa Malaysia atau Bahasa Inggerís.*

Baca arahan dengan teliti sebelum anda menjawab soalan.

Setiap soalan diperuntukkan 100 markah.

Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah digunapakai.]

1. (a) Show that the sequence $a_n = \dfrac{n}{n+2}$ is increasing.

(10 marks)

(b) Show that $\lim\limits_{x \to 0^+} x^{2x} = 1$.

(15 marks)

(c) Determine whether each of the following series converges or diverges.

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k(k+1)}$,

(ii) $\displaystyle\sum_{k=1}^{\infty} \frac{(2k)!}{k^2}$,

(iii) $\displaystyle\sum_{k=1}^{\infty} \frac{4^k}{(k+2)^k}$.

(30 marks)

(d) Find the interval of convergence of the power series

$$\sum_{k=1}^{\infty} \frac{x^k}{k5^k}.$$

(45 marks)

2. (a) Let $f(x,y) = \dfrac{x}{y} - \dfrac{y}{x}$. Find $f_{xx} + f_{xy}$.

(10 marks)

(b) Given

$$w = (x+y+z)^2, \qquad x = r - s, \qquad y = \sin(r+s), \qquad z = r^2.$$

Find $\dfrac{\partial w}{\partial s}$ when $r = 1$ and $s = -1$.

(20 marks)

(c) Given a function of two variables

$$f(x,y) = \sqrt{16 - x^2 - y^2}.$$

(i) Find the domain and range of the above function.

(ii) Sketch the level curves $f(x,y) = c$ corresponding to $c = 0, 2, 4$.

(iii) Hence sketch the graph of the function $f(x,y)$.

(30 marks)

(d) Given that

$$f(x,y) = x^3 + y^3 - 6xy.$$

Find all the local extreme values and saddle point, if any.

(40 marks)

3. (a) Show that

$$\lim_{(x,y)\to(0,0)} \frac{-xy}{x^2+y^2}$$

does not exist.

(20 marks)

(b) Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $(x+y)^2 = (y-z)^3$.

(20 marks)

(c) Suppose that a function $f(x,y,z)$ is differentiable at the point $(1,2,3)$ with $f_x(1,2,3)=3$, $f_y(1,2,3)=2$ and $f_z(1,2,3)=1$. If $f(1,2,3)=4$, estimate the value of $f(1.01,1.98,3.02)$.

(20 marks)

(d) Use the Lagrange multiplier method to find the maximum sum of $x^2+y^2+z^2$ if $x+2y+2z=12$.

(40 marks)

4. (a) Evaluate $\iint_R 6xy\,dA$ over the region R that is bounded by $y=x^2$, $x=2$ and $y=0$

(20 marks)

(b) Use the double integral to find the volume of a solid that is bounded by the cylinder $x^2+y^2=4$, the plane $y+z=4$ and the plane $z=0$.

(30 marks)

(c) Sketch the region of integration for the integral $\int_0^1\int_y^1 e^{x^2}\,dx\,dy$ and write an equivalent integral with the order of integration reversed. Hence, evaluate the integral.

(25 marks)

(d) Given the integral $\int_0^2\int_0^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$. Change the Cartesian integral into an equivalent polar integral. Hence, evaluate the integral.

(25 marks)

5. (a) Evaluate the integral

(i) $\int_0^1 \int_{-x}^{x} \int_0^{x+y} z \; dz\, dy\, dx$

(ii) $\int_0^{2\pi} \int_0^{\pi} \int_0^{\frac{1-\cos\phi}{2}} \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$.

(35 marks)

(b) Evaluate the integral

$$\iiint_T xyz \; dV$$

where $T = \{(x, y, z) : 1 \le x \le 3,\ -1 \le y \le 2,\ 0 \le z \le 2\}$.

(15 marks)

(c) Show that the volume of a sphere of radius a is $\frac{4}{3}\pi a^3$ unit^3 using the triple integral in spherical coordinates.

(25 marks)

(d) Use the triple integral in cylindrical coordinates to find the volume of a solid that is bounded by the paraboloid $z = x^2 + y^2$, the cylinder $x^2 + y^2 = 4$ and the plane $z = 0$.

(25 marks)

1. (a) Tunjukkan bahawa jujukan $a_n = \dfrac{n}{n+2}$ adalah menokok.

(10 markah)

(b) Tunjukkan bahawa $\operatorname*{had}_{x\to 0^+} x^{2x} = 1$.

(15 markah)

(c) Tentukan sama ada setiap siri berikut menumpu atau mencapah.

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k(k+1)}$,

(ii) $\displaystyle\sum_{k=1}^{\infty} \frac{(2k)!}{k^2}$,

(iii) $\displaystyle\sum_{k=1}^{\infty} \frac{4^k}{(k+2)^k}$.

(30 markah)

(d) Dapatkan selang penumpuan bagi siri kuasa

$$\sum_{k=1}^{\infty} \frac{x^k}{k5^k}.$$

(45 markah)

2. (a) Katakan $f(x,y) = \dfrac{x}{y} - \dfrac{y}{x}$. Cari $f_{xx} + f_{xy}$.

(10 markah)

(b) Diberi

$$w = (x+y+z)^2, \qquad x = r - s, \qquad y = \sin(r+s), \qquad z = r^2.$$

Cari $\dfrac{\partial w}{\partial s}$ apabila $r = 1$ dan $s = -1$.

(20 markah)

(c) Diberi suatu fungsi dua pembolehubah

$$f(x,y) = \sqrt{16 - x^2 - y^2}.$$

(i) Cari domain dan julat bagi fungsi di atas.

(ii) Lakarkan lengkung aras $f(x,y) = c$ untuk $c = 0, 2, 4$.

(iii) Seterusnya, lakarkan graf bagi fungsi $f(x,y)$.

(30 markah)

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(d) Diberi bahawa

$$f(x, y) = x^3 + y^3 - 6xy.$$

Cari semua nilai ekstremum setempat dan titik lengkok balas, jika ada.

(40 markah)

3. (a) Tunjukkan bahawa

$$\operatorname*{had}_{(x,\ y)\to(0,0)} \frac{-xy}{x^2 + y^2}$$

tidak wujud.

(20 markah)

(b) Gunakan pembezaan tersirat untuk mencari $\dfrac{\partial z}{\partial x}$ dan $\dfrac{\partial z}{\partial y}$ jika $(x+y)^2 = (y-z)^3$.

(20 markah)

(c) Katakan suatu fungsi $f(x, y, z)$ terbeza pada titik $(1,2,3)$ dengan $f_x(1,2,3) = 3$, $f_y(1,2,3) = 2$ dan $f_z(1,2,3) = 1$. Jika $f(1,2,3) = 4$, anggarkan nilai bagi $f(1.01, 1.98, 3.02)$.

(20 markah)

(d) Gunakan kaedah pendarab Lagrange untuk mencari jumlah maksimum bagi $x^2 + y^2 + z^2$ jika $x + 2y + 2z = 12$.

(40 markah)

4. (a) Nilaikan $\iint\limits_R 6xy\, dA$ ke atas rantau R yang dibatasi oleh $y = x^2$, $x = 2$ dan $y = 0$.

(20 markah)

(b) Gunakan kamiran ganda dua untuk mencari isipadu suatu bongkah yang dibatasi oleh silinder $x^2 + y^2 = 4$, satah $y + z = 4$ dan satah $z = 0$.

(30 markah)

(c) Lakarkan rantau kamiran bagi kamiran $\int_0^1 \int_y^1 e^{x^2} dx\, dy$ dan tuliskan kamiran yang setara dengan tertib kamiran disalingtukarkan. Seterusnya, nilaikan kamiran tersebut.

(25 markah)

(d) Diberi kamiran $\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2+y^2)\, dy\, dx$. Tukarkan kamiran Cartesian tersebut ke dalam kamiran kutub yang sepadan. Seterusnya, nilaikan kamiran tersebut.

(25 markah)

5. (a) Nilaikan kamiran

(i) $\int_0^1 \int_{-x}^{x} \int_0^{x+y} z\ dz\, dy\, dx$,

(ii) $\int_0^{2\pi} \int_0^{\pi} \int_0^{\frac{1-\cos\phi}{2}} \rho^2 \sin\phi\ d\rho\, d\phi\, d\theta$

(35 markah)

(b) Nilaikan kamiran

$$\iiint_T xyz\ dV$$

dengan $T = \{(x,y,z): 1 \le x \le 3,\ -1 \le y \le 2,\ 0 \le z \le 2\}$.

(15 markah)

(c) Tunjukkan bahawa isipadu suatu sfera yang berjejari a ialah $\frac{4}{3}\pi a^3$ unit^3 dengan menggunakan kamiran ganda tiga dalam koordinat sfera.

(25 markah)

(d) Gunakan kamiran ganda tiga dalam koordinat silinder untuk mencari isipadu suatu bongkah yang dibatasi oleh paraboloid $z = x^2 + y^2$, silinder $x^2 + y^2 = 4$ dan satah $z = 0$.

(25 markah)

List of formula:

1. $A = \frac{\partial^2 f}{\partial x^2}(x_0, y_0)$, $B = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$, $C = \frac{\partial^2 f}{\partial y^2}(x_0, y_0)$, $D = AC - B^2$

2. $\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}$, $\frac{\partial z}{\partial y} = -\frac{f_y}{f_z}$

3. $df = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z$

4. $\nabla f(x, y, z) = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k}$

5. $\nabla f(x, y) = \lambda \nabla g(x, y)$

6. $x = r\cos\theta$, $y = r\sin\theta$, $dA = r\,dr\,d\theta$

7. $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$, $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

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